Fractal scaling behavior of water flow patterns on inhomogeneous surfaces

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(Received 21 March 1996)

We have analyzed the patterns formed by water flowing over an inhomogeneous incline and have found fractal scaling behavior in the downhill direction, with d_f=1.35–1.45. The observed behavior is consistent with recent theoretical predictions involving nonlinear fluid flow close to threshold conditions. [S1063-651X(96)00212-7]

PACS number(s): 47.55.Mh, 68.35.Fx

I. INTRODUCTION

Stream patterns which form when water flows over an inhomogeneous surface are a commonly observed phenomenon in nature. Examples include the patterns which form on wet shower stalls, on windowpanes during rain storms, and on cool inclined surfaces when exposed to humidity. Such patterns consist of a hierarchy of streams which exhibit both splitting and joining, the net effect being a series of crossings which occur whenever two or more streams encounter each other as they make their way down the incline. The behavior is distinct from that of a river network [1–4] where, due to the effects of erosion, smaller tributaries flow into larger paths to form a treelike structure without crossings.

In contrast, the stream patterns of interest here form from water droplets which impact or condense onto the surface of a noneroding, inhomogeneous surface inclined at some angle. They remain in place until they acquire a critical mass such that the gravitational forces which pull the droplets downward are able to overcome the various forces (arising from surface tension effects, surface roughness, chemical inhomogeneities, etc.) which tend to pin the droplets in place. The droplets' motion, once initiated, does not proceed in a direct downward fashion on account of the surface irregularities which are encountered. Instead it is slowed, redirected, or stopped entirely depending on the nature of the irregularities encountered. As the total water mass increases, it must, however, eventually reach a threshold beyond which connected streams flow with finite velocity from top to bottom. The transition is in many ways reminiscent of avalanches [5] and depinning transitions which occur in a variety of systems [6].

We report here a measurement of the scaling properties of such a pattern for water flowing close to threshold conditions, and compare this result to recent theoretical predictions for fluid flow over randomly rough surfaces [7,8].

II. MEASUREMENTS AND IMAGE ANALYSIS PROCEDURE

The particular stream patterns which we studied were formed by water flow over a dirty glass surface during a rain storm. Although in principle some erosion (''washing'') of the surface contaminants occurred during the measurements, we did not observe this effect for the surfaces we selected to study, the planar rear windshields of unwashed cars. Such surfaces are inhomogeneous, presenting the water droplets with a random distribution of irregularities and pinning sites as they make their way down the surface. A photograph of a typical rain pattern is depicted in Fig. 1.

Figures 2(a) and 2(b) present two digitized images of typical patterns photographed in light rain conditions. Stream crossings are common in the left-hand image and less so in the right-hand image. In both images, however, a number of continuous or flowing (spanning top to bottom) and isolated streams are in evidence. In addition there are channels which flow into streams, which we refer to as tributaries (see Fig. 3). The tributaries feed those streams which span the top to bottom. We note that the average stream width varies little from top to bottom, in direct contrast with river networks where the thinner streams flow into thicker ones [2].

We used four different methods to calculate the fractal dimension d_f [6,9,10] of the stream patterns.

Method A. By covering the structure with boxes of different sizes ϵ, with $\epsilon\to 0$, the fractal dimension d_f is obtained as

$$N(\epsilon)\sim\epsilon^{-d_f}, \tag{1}$$

where N is the number of boxes of size ϵ needed to cover the water patterns.

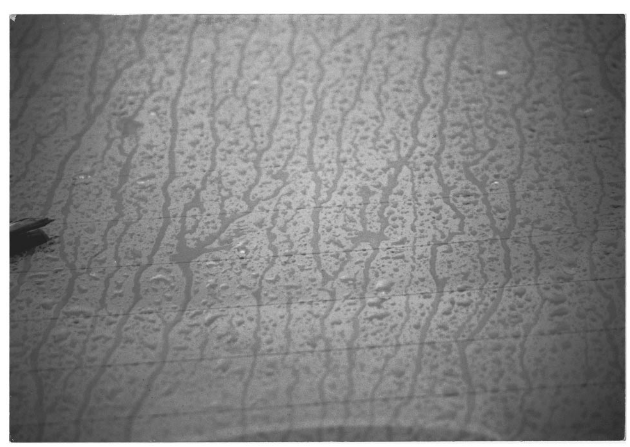

FIG. 1. Photograph of a typical rain pattern taken in light rain conditions.

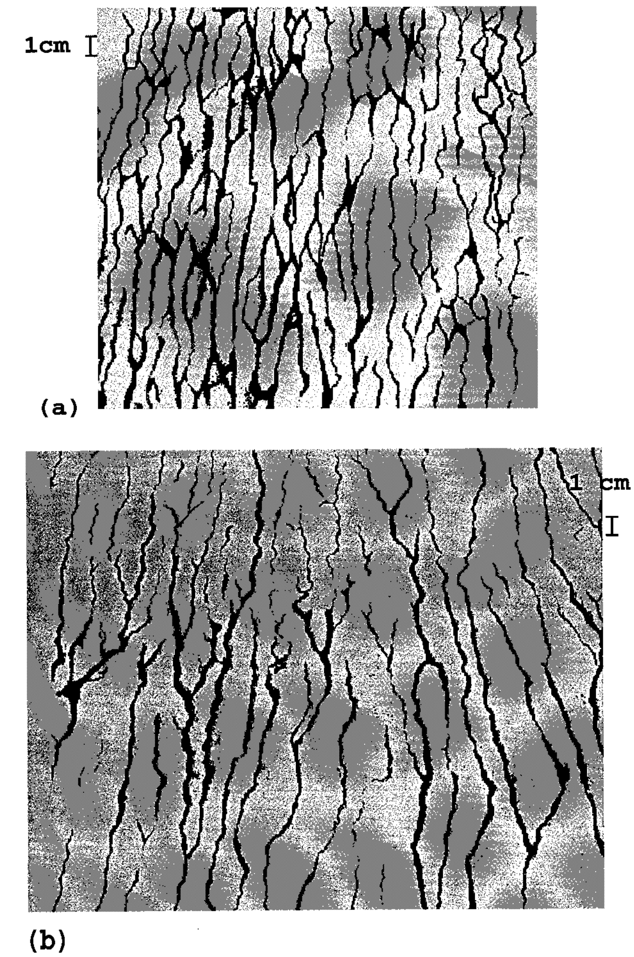

FIG. 2. (a) and (b) Two examples of digitized images of rain patterns. Crossings are observed as well as many flowing streams, isolated clusters, and tributaries. The size of the picture does not allow one to see the total length of flowing streams. The first image has real dimensions: 25.3 cm×23.9 cm.

Method B. The mass of a particular stream m scales with its downhill length l as

$$m(l) \sim l^{d_f}. \tag{2}$$

In this method many streams are analyzed, and each one is represented by a curve from which the fractal dimension is calculated. An average over all curves yields the final value of d_f.

Method C. The total mass of the streams M whose total length is L scales as

$$M(L) \sim L^{d_f}. \tag{3}$$

In this method each stream in the plot is represented by a point instead of a curve. After plotting on a log-log graph, these points yield a straight line whose slope is the fractal dimension.

Method D. The last method considered is the pair correlation function method [9] defined as

$$c(\vec{r}) \equiv N^{-1} \sum_{\vec{r}_0} \rho(\vec{r}_0)\rho(\vec{r}_0+\vec{r}). \tag{4}$$

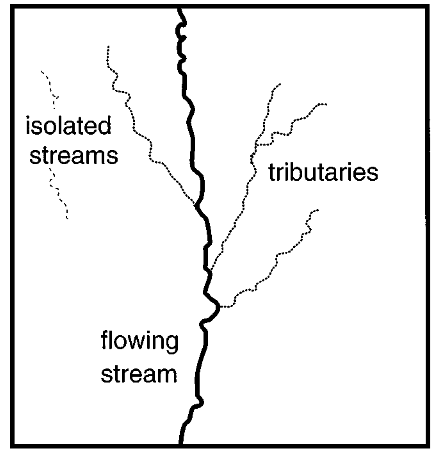

FIG. 3. Schematic representation of the tributaries and isolated and flowing streams.

Here, ρ is the local density defined as $\rho(\vec{r})=1$ if the point at position $\vec{r}$ belongs to the structure and $\rho(\vec{r})=0$ otherwise, and N is the total number of points used to calculate the correlation function. $c(\vec{r})$ represents the expectation value that two points separated by a distance $\vec{r}$ belong to the pattern.

Isotropic fractal structures are characterized by a correlation function which does not depend on the direction of $\vec{r}$, so that $c(\vec{r})=c(r)$. For the case at hand, however, the vertical and horizontal directions are anisotropic. We therefore take $\vec{r}$ parallel to the downhill direction (i.e., $|\vec{r}|=l$).

The correlation function satisfies a power law of the form

$$c(l) \sim l^{-\alpha}. \tag{5}$$

The mass of a stream $m(l)$ can be expressed in terms of the correlation function $c(l)$ from which it follows that the fractal dimension is $d_f=2-\alpha$ [9].

When using method A we took into account the small distortion of the picture due to the fact that the photograph was taken at a slight angle with respect to the surface normal, and we superimposed trapezoidal (instead of the usual squares) grids onto this figure with various mesh sizes ϵ, where ϵ is proportional to the inverse of the area of the total grid that covers the entire structure.

For the other three listed methods, we digitized the pictures with an Apple Scanner Model 16 with resolution 70 dots per inch. The scanner yields a matrix whose elements are 1 and 0, giving 1 when the point belongs to the water structure and 0 when it does not. We used a ''burning'' algorithm [11] to compute the mass of each isolated stream, and obtained the fractal dimension using methods B and C. In each of these methods, the mass of the stream is the number of sites in it given by the output matrix of the Apple Scanner. Typical digitized images are shown in Fig. 2.

III. RESULTS

We calculated d_f for (i) individual tributaries and isolated streams and (ii) the entire pattern, i.e., not only tributaries and isolated streams but also streams that span the entire image from top to bottom.

(i) Tributaries and isolated streams. To calculate the

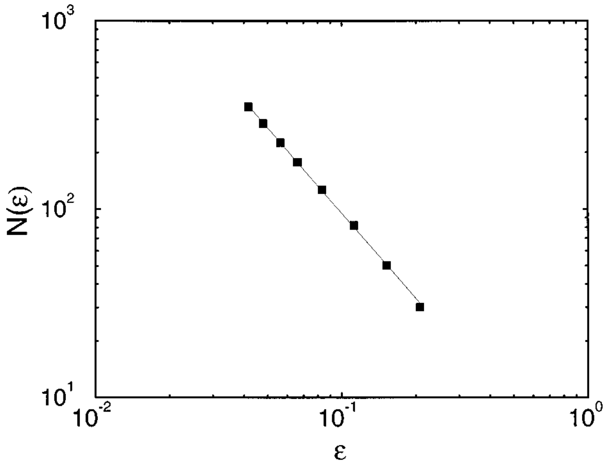

FIG. 4. Result of method A. Number of trapezoids $N(\epsilon)$ needed to cover the water structure of Fig. 2(a) in terms of their size ϵ for the tributaries and isolated streams. The slope has an absolute value of 1.38±0.03.

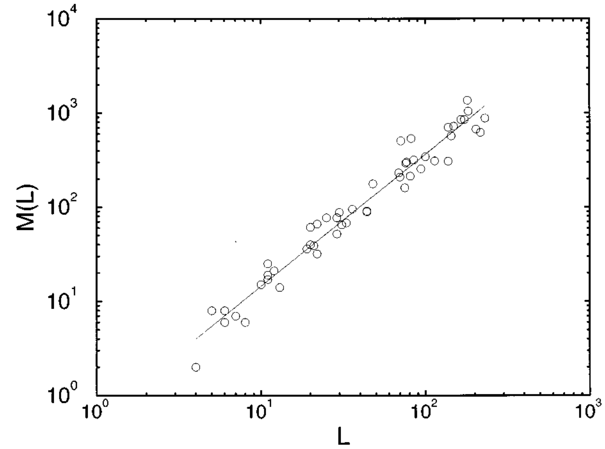

FIG. 6. Total mass M as a function of total downhill length L for the average of tributaries and isolated streams (method C). The slope obtained for this curve gives a fractal dimension d_f=1.40±0.02.

fractal dimension of these structures, we removed the streams spanning top to bottom in all of the digitized images of the windshields. Figure 7 shows one of the images obtained, after having done so. We applied methods A, B, and C listed above to calculate d_f, and these results are shown, respectively, in Figs. 4–6. We have averaged over six different windshields for methods B and C whereas we applied method A to only two water patterns, one of which is shown in Fig. 7.

In the present case we also used method D to calculate d_f. However, due to the small length of the tributaries and isolated streams, the scaling only extended to less than one decade, and the results were not reliable.

(ii) Entire pattern: all tributaries, isolated streams, and streams spanning top to bottom. We applied method A only to Fig. 2(b), and the results are shown in Fig. 8. We also used methods B, C, and D to calculate d_f, averaging over six different windshields.

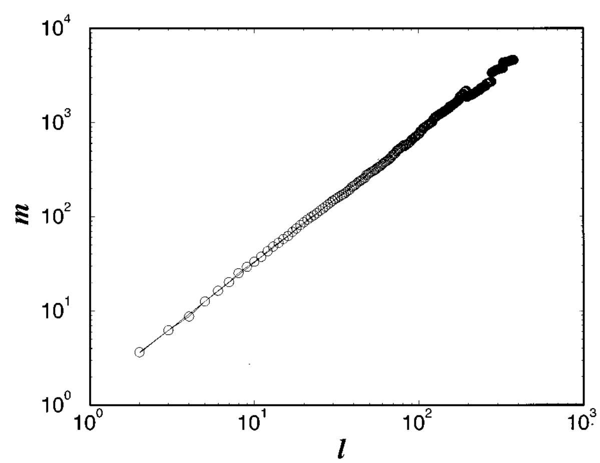

FIG. 5. Plot of the mass of tributaries and isolated streams as a function of its downhill length l (method B) for a particular water pattern. The average of all the individual d_f yields a value of 1.34 ±0.03.

In Fig. 9 we observe the average of all the curves representing the mass of each individual stream versus its downhill length for one particular rain pattern [i.e., this picture (method B) is an average over all the streams that form one of the entire water patterns].

Figure 10(a) shows the total mass M versus the total length L for the entire water pattern of Fig. 2(b) (method C). Results from the pair correlation function method for the entire water pattern are plotted in Fig. 10(b) according to the scaling relation of Eq. (5). We see a good scaling over two decades. At the extremes, saturation effects due to the size of the picture are observed.

The results are summarized in Table I.

IV. DISCUSSION

Narayan and Fisher (NF) [7] have recently published a model for fluid flow which is applicable to a system which is very similar to that investigated here. They consider the case of a rough, random surface upon which a fixed amount of

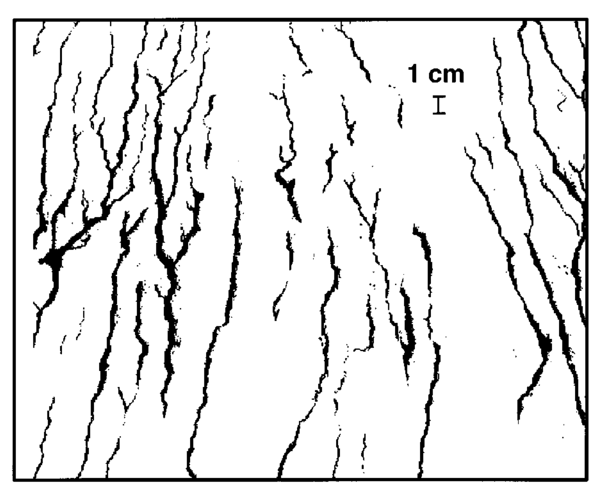

FIG. 7. Tributaries and isolated streams obtained from Fig. 2 by removing the streams that flow from top to bottom.

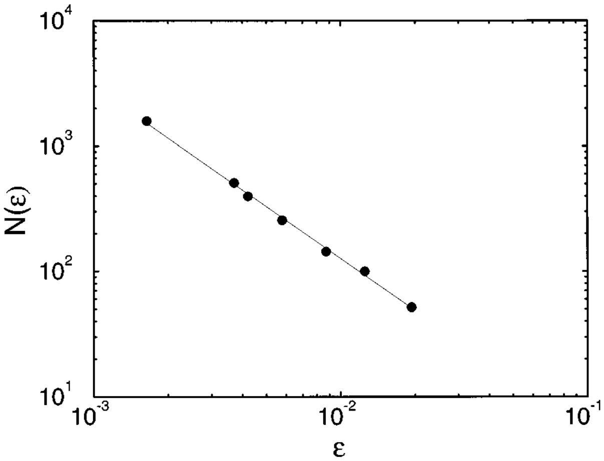

FIG. 8. Result of method A. Number of trapezoids $N(\epsilon)$ needed to cover the water structures of Fig. 2(b) in terms of their size ϵ for the entire water pattern. The slope has an absolute value of 1.46 ± 0.05.

fluid is initially placed. As the surface's angle of incline is slowly increased, the fluid collects into lakes with a certain depth, which, when saturated, fill the adjacent ones underneath, forming clusters of lakes. If the angle of incline is small, then the water does not fill the lakes completely. When the angle of incline is larger, the length of the clusters increases, reaching an infinite value at a critical threshold angle. Below this limit, it is possible to see many isolated clusters which are totally disconnected from the flow. At threshold, which is the equivalent of the depinning transition, there exists at least one flowing river from top to bottom (e.g., at least one cluster whose correlation length is infinite). Above threshold one may distinguish more than one flowing river, and far above threshold the flow is uniform, it being difficult to distinguish any individual rivers.

For close to threshold conditions, NF found the value for the fractal dimension in the downward direction to be 1.21 ± 0.02 by numerical simulations of their model, and $\frac{4}{3}$ by mean field theory. We consider the rain patterns discussed

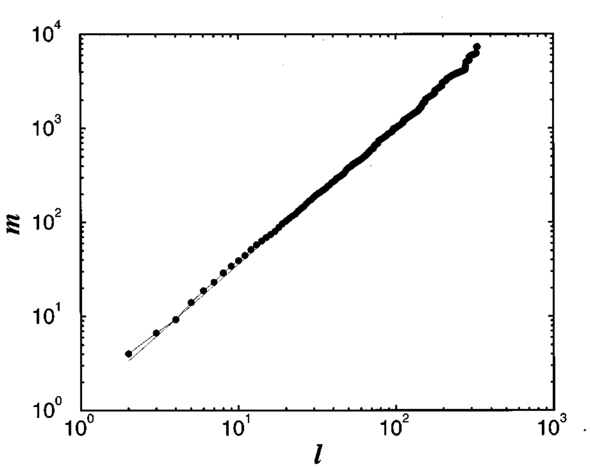

FIG. 9. Mass of each individual stream m vs its downhill length l for one of the entire water patterns (method B). The slope for this plot yields a fractal dimension of 1.47 ± 0.01.

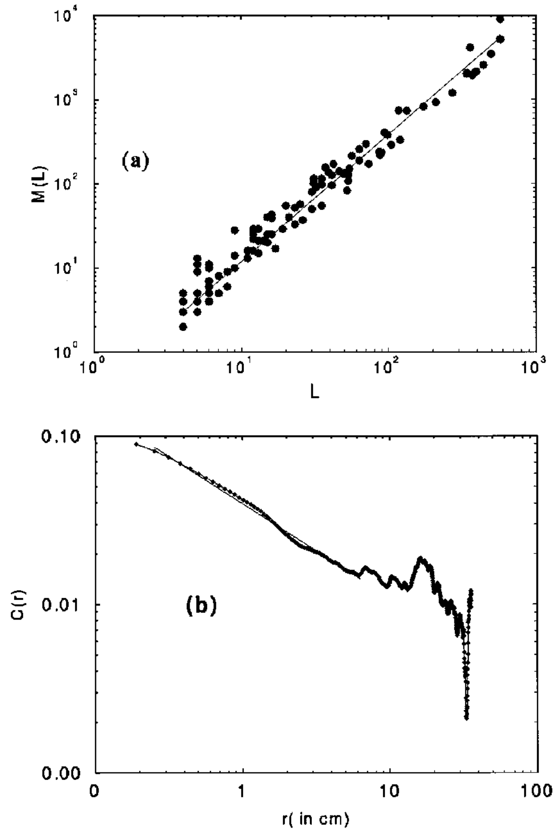

FIG. 10. Total mass M as a function of total length L for the entire water pattern, (a) using method C. The data plotted fitted to a straight line, showing fractal scaling behavior with $d_f = 1.51 \pm 0.02$. (b) Pair correlation function $C(r)$ vs r using method D for one of the entire water patterns. The slope of the curve is -0.56 ± 0.05, yielding a fractal dimension of $d_f = 1.44 \pm 0.02$.

here to be above threshold, since they all contained at least one stream which spanned the image from top to bottom. The patterns must be close to threshold, however, since otherwise it would be very difficult to distinguish individual streams, i.e., the water would be flowing in a sheet, with little spatial separation.

The NF model is similar, but not identical to the fluid flow studied here, since it deposits a fixed initial amount of liquid onto a surface rather than continously raining fluid down upon a surface tilted at a fixed angle. Since the fluid mass is fixed, the NF model requires a varying tilt angle in order to provide a force to drive the fluid flow. Watson and Fisher (WF) [8] have very recently considered a model which is more similar to the case studied here. In this model the particles move downhill on a square lattice, and the driving is

TABLE I. Fractal dimension d_f.

Method	Tributaries and isolated streams	Entire pattern
A	1.38 ± 0.03	1.46 ± 0.05
B	1.34 ± 0.04	1.44 ± 0.02
C	1.4 ± 0.02	1.4 ± 0.03
D		1.44 ± 0.01

increased by adding particles to some fraction of the sites. The WF model is stochastic, with a random capacity for each site and random local rule determining where particles will move when a given site overflows. WF argue that the statistics of the model should be in the same universality class as the NF model, and, with slightly different boundary conditions, it would be applicable to water flow over dirty windshields.

The value for d_f which we observe for tributaries and isolated clusters (which presumably have the same fractal dimension above and below threshold) should therefore be comparable to the fractal dimension predicted by the NF and WF models. Our observed value $d_f = 1.37 \pm 0.05$, while distinctly lower than that typical of actual river networks subjected to erosion ($d_f = 1.6$–1.7) [1], compares favorably with the NF and WF prediction of $\frac{4}{3}$ by mean field theory within the error of the measurement.

ACKNOWLEDGMENTS

One of the authors (M.S.T.) would like to thank H. Makse for very useful discussions. This work has been supported by NSF Grant No. DMR9204022 and PRF Grant No. 27498-AC.

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